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Quantum Versus Stochastic Processes and the Role of Complex Numbers[1]

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We argue that the complex numbers are an irreducible object of quantum probability: this can be seen in the measurements of geometric phases that have no classical probabilistic analogue. Having the complex phases as primitive ingredient implies that we need to accept nonadditive probabilities. This has the desirable consequence of removing constraints of standard theorems about the possibility of describing quantum theory with commutative variables. Motivated by the formalism of consistent histories and keeping an analogy with the theory of stochastic processes, we develop a (statistical) theory of quantum processes: they are characterized by the introduction of a "density matrix" on phase space paths (it thus includes phase information) and fully reproduces quantum mechanical predictions. We can write quantum differential equations (in analogy to Langevin equation) that could be interpreted as referring to individual quantum systems. We describe the reconstruction theorem by which a quantum process can yield the standard Hilbert space structure if the Markov property is imposed. We discuss the relevance of our results for the interpretation of quantum theory (a sample space is possible if probabilities are nonadditive) and quantum gravity (the Hilbert space arises here after the consideration of a background causal structure).

KEY WORDS: quantum probability; geometric phases; consistent histories; quantum processes.

1. INTRODUCTION

. . . Is what the matrix-physicists and q-number-physicists say true—that the wave equation describes only the behavior of a statistical ensemble, just like the so-called Fokker differential equation? . . .

Schrödinger to Planck, 1927

This quotation is from a letter Schrödinger sent to Planck (Przibram, 1967) after the fifth Solvay Congress in 1926. In this congress the matrix mechanics of Heisenberg and the wave mechanics of Schrödinger had faced each other and were

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reconciled by the brilliant idea of Born to interpret Schrödinger's wave function as corresponding to probability.

This proved to be one of the last pieces in the puzzle towards the development of a full-fledged quantum theory. But Schrödinger was not satisfied with this interpretation; he distrusted the philosophy underlying the physical ideas of Heisenberg, Born, and Bohr ("the matrix-physicists and q-number-physicists" in the quotation). In fact, he wanted to interpret the wave function as a physical wave, but he could not: the wave function was defined on the configuration space and not on the physical space, as any physical wave ought to.

Fokker's equation, to which Schrödinger refers is known today as the Fokker–Plack equation. It is the equation that describes the time evolution of the probability density in Brownian motion. For Schrödinger it was natural to consider that a probabilistic interpretation of the wave function would amount to its having a physical function analogou's to that of a classical probability distribution. In such a case one could say that the wave function provides the description of a statistical ensemble rather than an individual quantum system.

But the interpretation of Schrödinger's equation as a Fokker–Planck type of equation did not gain ground and for good reason. The structure of these equations are very distinct. The Fokker–Planck equation reads

$$\frac{\partial}{\partial t}\rho = \mathcal{L}\rho, \tag{1.1}$$

where ρ is a probability distribution and $\mathcal{L}$ a linear operator in the space, where the ρ live. Schrödinger's equation is of course

$$i\frac{\partial}{\partial t}\psi = H\psi \tag{1.2}$$

H is the Hamiltonian, a self-adjoint operator on the Hilbert space, where the ψ live. The difference between these two equations and what distinguishes between classical and quantum probability is the presence of the complex unit i. Well, you might argue that complex numbers are not really measurable quantities, just a convenient device to simplify the writing of equations; indeed, Schrödinger's equation can be written without the use of i by splitting it into two equations, one for the real and one for the imaginary part of ψ. However, physical observables arise out of the probability distribution $p = |\psi|^2$, so even if we forego the use of complex numbers they will reappear in the form of the $U(1)$ invariance of the probabilities.

In effect, what distinguishes quantum from classical probability is the presence of the complex numbers, not so much in the dynamics, but as the $U(1)$ symmetry of the probability assignment.

Coming back to the Fokker–Planck equation, we note that it is a part of the more general theory of stochastic processes. The stochastic processes can describe

the dynamics in two ways: either through the Fokker–Planck equation by writing a deterministic equation for the probability density that refers to an ensemble of physical systems, or through stochastic differential equations (like the Langevin equation) that can be interpreted as referring to the behavior of an *individual* system evolving under some random forces. In classical probability, we then have the following diagram

$$\text{Fokker–Planck equation} \leftarrow \textit{stochastic processes} \rightarrow \text{Langevin equation}$$

If we try to interpret Schrödinger's equation as analogous to the Fokker–Planck equation, we immediately see that we lack both the general theory in which it will be embedded and the description (analogous to differential equations) that could be interpreted as referring to individual quantum systems. So the corresponding diagram has two gaps in it

$$\text{Schrödinger's equation} \leftarrow X_1 \text{ theory} \rightarrow X_2 \text{ equation}$$

Nelson attempted to fill the gap in his stochastic mechanics programme (Ghirardi *et al.*, 1978; Nelson, 1985), where for X_1 he considered again the theory of stochastic processes and for X_2 again stochastic differential equations.[3] However, this description cannot account for all quantum phenomena as it violates Bell's theorem. In fact, no stochastic process can reproduce all quantum mechanical predictions (Grabert *et al.*, 1979).

In this talk we are going to show how to fill in the previous diagram, without diverging from the predictions of standard quantum theory; we are going to write a class of theories which is modelled on stochastic processes but are not stochastic processes themselves, since they intrinsically incorporate the appearance of the complex numbers (or if you prefer a $U(1)$ symmetry). This class of theories we shall call *quantum processes*. They have been developed in a chain of argumentation starting from the consistent histories approach to quantum theory and the presence of the geometric phase. Quantum processes is the X_1 in our previous diagram; they can be unravelled to write full analogues of the stochastic differential equations, which could perhaps be interpreted as referring to individual system.

In general, our approach is very close to the ideas about quantum theory, that Einstein expressed in his later years (Einstein, 1975) (in which it was made clear that his disagreement with quantum theory was not on reasons of determinism, but on reasons of realism)

> ... ... The concept that the ψ-function completely describes the physical behaviour of the individual single system is untenable. (...) But if one regards the ψ-function as the description of an ensemble, it furnishes statements that correspond satisfactorily to those of classical mechanics and at the same time account for the quantum structure of reality. ... ...

[3] He actually considered the Madelung equations that are derived from Schrödinger's equation and the probability rule.

In this paper, we demonstrate that this attitude is not forbidden by any non-go theorem of quantum mechanics; indeed at the mathematical level we can even write equations that could be interpreted as referring to individual systems. And, at least for the author, this attitude is in no way incompatible with the basic physical insights, that were put into the structure of quantum theory by Heisenberg and Bohr.

2. CLASSICAL VERSUS QUANTUM PROBABILITY

Quantum theory was built in the late twenties and by the early thirties its basic principles and structures had been put in place. During the same period, classical probability was set in a solid axiomatic framework in the work of Kolmogorov. Kolmogorov founded probabilistic calculus on measure theory and thus removed ambiguities plaguing other intuitive approaches to probability. In particular, measure theory provided a framework, by which the theory of stochastic processes could be rigorously developed.

This was arguably one of the most important results in twentieth century's applied mathematics; in particular, it influenced the more mathematically-minded people that worked on the foundations of quantum theory. In particular, it was soon apparent that probability theory and quantum mechanics seemed to share the same basic "physical" concepts, even if their mathematical implementation is distinct. Classical probability theory is defined on a sample space Ω, which is an ordinary set (often a manifold), while quantum theory is defined on a *complex* Hilbert space H.

A theory such as classical probability is a mathematical framework by which physical phenomena can be modeled, so that the statistical results (mean values, ratios of events) measurements can be predicted. So is quantum theory, at least in the Kopenhagen interpretation. Both theories employ the notions of

1. *Observables*: An observable is a physical quantity, whose value we measure. It is assumed that what we observe can only be a real number, since it is with respect to real numbers (distances in dials) that we encode all experimental information. It has to be a single-valued object, so that when we have a concrete measurement situation there will be no ambiguity as to the quantity we measure. Classically an observable is a measurable function on Ω, while quantum mechanically a self-adjoint operator on H.
2. *Events*: As events we characterize the possible outcomes of individual experiments: an event corresponds to a property that is verified by an experiment, e.g. the particle passed through a slit located in that particular place. Quantum mechanical events are represented by projection operators and classical ones by measurable subsets of the sample space.
3. *States*: A state corresponds to the preparation of the physical system, before the measurement is carried out. As such, it contains information as much

about the nature of the physical system as about the preparation procedure that was used for the experiment. A state should be a mathematical object that would be able to provide all information that can be accessed experimentally. If we think that in experiments we can determine mean values for observables and (perhaps) probabilities to events, the state should be an object that provides these. In classical probability theory a state corresponds to a probability distribution, while in quantum theory (thanks to Gleason's theorem) by a density matrix. Note, however, that in representing the operational notion of the state by such mathematical objects we assumed that the probabilities that correspond to physical systems satisfy a number of properties. Most important of them is the additivity condition, that if A and B are two independent events then $p(A \cup B) = p(A) + p(B)$. This is a theoretical "prejudice" rather than an axiom that arises naturally out of the consideration of the measurement processes.

In any case, starting from the work of von Neumann, Birkhoff, Wigner, and Jordan in the thirties (Birkhoff and von Neumann, 1936; Jordan *et al.*, 1934), an ever increasing number of mathematical physicists considered quantum theory to be nothing but a generalized probability theory, i.e. a theory sharing the same basic objects as classical probability theory. This attitude is apparent in many schemes of axiomatisation of quantum theory, that want to get rid of the (mathematically) unintuitive structure of a Hilbert space (while keeping the noncommutativity of observables), by substituting it with something simpler: algebras of observables in the C^*-algebraic approach, lattice of propositions in quantum logic(s), convex state spaces in the operational approach.

However, we still feel the need to ask the question, whether *quantum theory is nothing but a generalized probability theory*.

Our answer here is that *it is not*.

The analogy between classical probability and quantum mechanics stops, when we consider properties of the quantum system at more than one moment of time. Because in this case

1. Interference phases appear that have no analogue in classical probability theory. They are closely related to the geometric phases. Most important of all, they are measured as a statistical object.
2. The probabilities for properties defined on more than one moment of time are *nonadditive*.
3. The interference phases and the nonadditive probabilities are closely related.
4. The natural correlation functions of the observables are probabilities generically complex-valued.

It seems that in the case of studying properties at more than one moment of time, the complex numbers inherent in the structure of the quantum mechanical Hilbert, space become manifest.

2.1. Nonadditive Probabilities

To explain the nonadditive probabilities in quantum theory let us consider the following type of experiment. We have a source S, which emits some particles (our individual quantum systems) prepared in a well-defined state $|\psi\rangle$. The ensemble of systems is then represented by a beam; we let this beam cross two filters represented by P and Q. A filter is an object that lets particles of the beam pass if they satisfy a certain property or, in other words, if a particular event occurs. In quantum theory a filter is represented by a projection operator, hence P and Q are projectors.

Now, after the second filter Q the beam falls into a detector D and we can measure the particles that have crossed the beams. Note that in the diagram the distance between the source, the filters and the detector is assumed to represent time intervals.

If the source emitted N particles and the detector detected n, then for large N the ratio n/N ought to converge to a given number, which would be the probability for the particles prepared in the state $|\psi\rangle$ to pass through the two filters.

The rules of quantum theory give that this probability must be equal to

$$p(P, t_1; Q, t_2) = \langle\psi|P(t_1)Q(t_2)P(t_1)|\psi\rangle$$
$$P(t) := e^{iHt} P e^{-iHt}, \tag{2.1}$$

where H is the Hamiltonian that describes the self-dynamics of the individual system (see Fig. 1).

Now consider the following three experiments. In experiment number 1, we put in as a first filter P_1 and as second one Q. In experiment number 2, we put as a first filter $P_2 = 1 - P_1$, i.e. the filter corresponding to the property complementary of that of P_1 and keep Q as a second filter. In experiment number 3, we simply have filter Q. If we use the rule (2.1) for the measured probabilities we see that

$$p_3 \neq p_1 + p_2. \tag{2.2}$$

This implies that the quantum mechanical probabilities do not satisfy the additivity condition. This simple experiment is a manifestation of the more general fact for

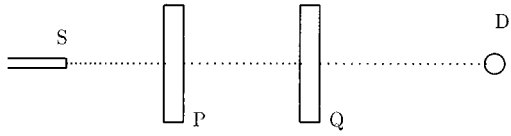

Fig. 1. Measurement of probabilities for histories.

quantum theory: *experiments involving properties of the system at more than one moment of time cannot, in general, be modeled by an additive probability measure*. So quantum probability for histories does not satisfy the basic Kolmogorov axioms for probabilities.

2.2. Consistent Histories

Well, why would we mind if the probabilities are nonadditive? Nonadditivity of implies you lose an important rule of inference. Consider that you have two exclusive events A and B, that are also exhaustive (one or the other can happen and nothing else). If we measure the probability for A and we find $p(A) = 1$, then we can be certain that the event B would never occur in any repetition of this experiment, since $p(B)$ has to equal zero. If probabilities are nonadditive, then the fact that $p(A) = 1$ does not imply that $p(B) = 0$ and the event B would take place in the experiments with nonzero frequency.

In a nutshell, even if the probability of an event is 1, we cannot preclude that its complement will never happen. Is this so bad? Not really if we have an operational stance, that quantum theory describes experiments in ensembles. However, it could be problematic if one wants to claim that the present formalism of quantum theory provides a theory for individual closed systems, because it would dramatically limit its predictability.

One way to resolve this problem (assuming it is a problem) is the consistent histories approach. This was developed by Griffiths (1984), Omnés (1988, 1992, 1994), Gell-Mann and Hartle (1990, 1993; Hartle, 1993). This work is motivated from its elegant formulation that has been developed by Isham (1994; Isham *et al.*, 1998; Isham and Linden, 1994, 1995) and Savvidou (1999, 2002).

As far as this issue goes, the key idea of the consistent histories approach is that one can have additive probabilities if one is restricted within particular sets of histories, known as *consistent sets*. More precisely

1. A general history α is represented by a collection of projection operators α_{t_i} at successive instants of time

$$\alpha = \left(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}\right), . \tag{2.3}$$

2. From these operators we can define an operator $C_\alpha = \alpha_{t_n}(t_n) \ldots \alpha_{t_2}(t_2)\alpha_{t_1}(t_1)$ and a complex-valued functional on pairs of histories

$$d(\alpha, \beta) = Tr(C_\alpha \rho_0 C_\beta^\dagger). \tag{2.4}$$

   This is known as the decoherence functional.
3. If in a exhaustive and exclusive set of histories

$$d(\alpha, \beta) = 0, \quad \alpha \neq \beta \tag{2.5}$$

then $d(\alpha, \alpha)$ is a probability for the history α and corresponds to an additive probability measure. The satisfaction of condition (2.5) renders a set of histories consistent.

The additivity of probabilities in each consistent set allows us to use the fact that probability one for an event means probability zero for its complement. This can be exploited to define the logical/physical notion for implication. The idea is that if we get by measurement a definite result for an *individual* system, we can employ the definability of this implication to identify other properties that this system has satisfied. However, implication is defined only within given consistent sets; hence when we employ implication in different consistent sets we can derive contrary results from the same definite (measured) event. This is a more general problem of realist interpretational schemes[4] and is related to the Kochen–Specker theorem.

However, the *formalism* of consistent histories makes sense irrespective of the interpretation: it can equally well be considered in a Copenhagen framework, in which we are content to provide predictions for outcomes of ensemble measurements. The merit of the formalism lies in the fact that it allows the description of quantum systems using *covariant objects* (histories) (Hartle, 1993). As such it seems more adequate to deal with basic problems in the quantization of the gravitational field, such as the problem of time (see Savvidou, 2001, for this perspective). Moreover, the object that was introduced by the consistent histories approach, the decoherence functional is very convenient: it will be shown to contain the full information that can be extracted from a quantum process, even its nonprobabilistic aspects.

2.3. Complex-Valued Correlation Functions

Whenever we have a probabilistic system, there is a well defined prescription by which the *temporal correlation functions* can be determined. Assume we have an observable A, that can be spectrally analysed as $A = \sum_i a_i P_i$, where P_i are filters that can be experimentally employed. Now we repeat the experiment in Fig. 1 with P_i for P and P_j for Q for all possible combinations of i and j. We can then measure the probabilities $p(i, t_1; j, t_2)$.

The statistical correlation function is then

$$\langle A_{t_1} A_{t_2} \rangle s = \sum_{ij} a_i a_j \, p(i, t_1; j, t_2). \tag{2.6}$$

It is clearly a real number. However, it is an object that cannot be naturally written in terms of the operators A. Moreover, it depends sensitively in the resolution of

[4] In our use, we employ the word realist to denote the attitude that the quantum mechanical formalism refers to properties of individual systems.

the observable in terms of the spectral projections (unlike statistical correlation functions in classical probability theory).

From quantum theory the natural object for the correlation function (that does not depend on the spectral resolution of the observable) is

$$\left\langle A_{t_1} A_{t_2} \right\rangle Q = \langle \psi(t_1) | A e^{iH(t_2-t_1)} A e^{-iH(t_2-t_1)} | \psi(t_1) \rangle, \tag{2.7}$$

or the corresponding time-ordered function.

This is generically a complex-valued object and has no natural operational interpretation as the statistical correlation function.

We should remark that temporal correlation functions have actually been measured in quantum optics (see for instance, Walls and Milburn, 1994). In that case the relevant observable is the photon number; this, however, commutes with the electromagnetic field's Hamiltonian. This implies that the statistical and quantum correlation functions coincide.

So one may pose the question, *why does quantum theory give as natural correlation functions, ones that are not operationally implementable?* This is not a very sharp question, as the answer might be simply that there is no a priori reason to expect that it would be otherwise. However, we are going to show that there is a deeper reason and this is that quantum correlation functions have information from measurable quantities that do not correspond to probabilities: *interference phases*.

3. INTERFERENCE PHASES

The point is that quantum theory predicts other physical quantities that can be determined statistically, but are not probabilities. These are the quantum phases and more precisely the geometric phases, paradigmatic example of which are the Bohm–Aharonov (Aharonov and Bohm, 1959) and the Berry phase (Berry, 1984).

Let us recall the measurement of the Bohm–Aharonov phase (see Fig. 2).

The basic configuration is that of a two-slit experiment. We let a prepared state of electrons cross through two slits and then measure the interference pattern on a screen. Having stored that in memory, we repeat the experiment by putting a solenoid (with some magnetic flux) behind one of the slits (such that the beam does not cross it). We observe a shift into the interference pattern, which is essentially proportional to the Bohm–Aharonov phase induced by the magnetic flux.

Two remarks must be made for this experiment, that are valid for all experiments measuring geometric phases. First, that the Bohm–Aharonov phase is a *statistical* object: it is measured in terms of an interference pattern, which is present only when a large number of electrons (thought of as corresponding to a statistical ensemble) are left to interfere. If we carried out the experiment with a single electron, there would be nothing to measure.

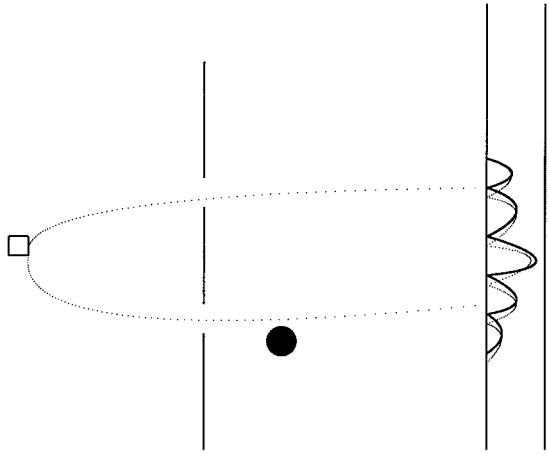

Fig. 2. Determination of the Bohm–Aharonov phase.

The second remark is that the geometric phase can be determined from the study of the interference of two beams with different history. This is reminiscent of the decoherence functional, which assigns a complex-valued object to a pair of histories. This suggests that the decoherence functional is somehow related to the geometric phases. This loose connection will turn out to be very precise: we can show that the decoherence functional is actually constructed from the mathematical object that is responsible for the presence of geometric phases: the Berry connection[5] (Anastopoulos and Savvidou, 2002).

More strongly, we can explicitly describe, how the off-diagonal elements of the decoherence functional can be explicitly measured; the corresponding measurement scheme is identical to the ones used for the determination of another version of the geometric phase, known as the *Pancharatnam* phase (Pancharatnam, 1956; Samuel and Bhandari, 1988). This phase essentially corresponds to the argument of the inner product between two states $\phi\rangle$ and $|\psi\rangle$. It is manifested in the following generic situations

1. Prepare two systems in the states $|\psi\rangle$ and $|\phi\rangle$.
2. Perform on the beam corresponding to $|\psi\rangle$ the operation $|\psi\rangle \to e^{i\chi}|\psi\rangle$ for a controlled value of χ.
3. Interfere the two beams to construct the beam $|f\rangle = |\phi\rangle + e^{i\chi}|\psi\rangle$ and measure its intensity $I = \langle f|f\rangle$.
4. Repeat the experiment for the range of all values of χ and construct the function $I(\chi)$ giving the intensity of the measured beam as a function of

[5] This is a $U(1)$ connection on the fiber bundle with base space the projective Hilbert space and total space the Hilbert space of a quantum system (Aharonov and Anandan, 1987; Simon, 1983).

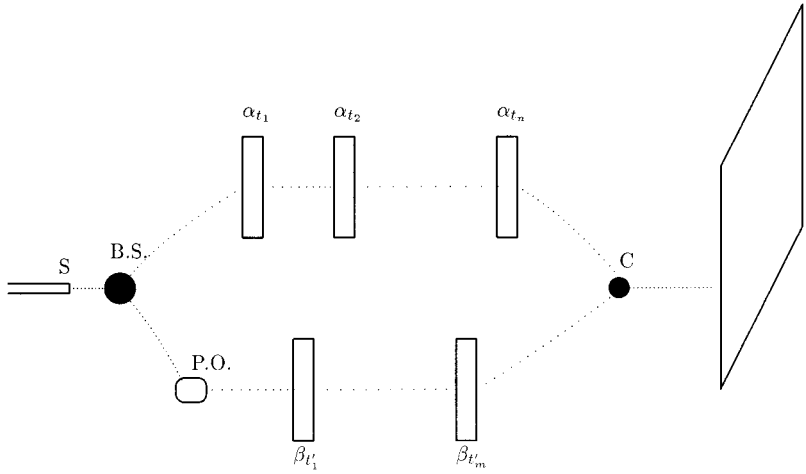

Fig. 3. Measurement of the off-diagonal elements of the decoherence functional.

the external parameter χ. This equals

$$I(\chi) = 2|\langle\psi|\phi\rangle| \cos(\chi - \arg\langle\psi|\phi\rangle) \tag{3.1}$$

5. $I(\chi)$ takes its maximum value for $\chi = \arg\langle\psi|\phi\rangle$. This value of χ is the Pancharatnam phase between the two beams.

This procedure to measure the Pancharatnam phase has been performed in neutron interferometry (Wagh *et al.*, 1998). The difficult part is to perform step 2, i.e. to have a controlled way to change the phase of an individual quantum state. This can be achieved if $|\psi\rangle$ is an eigenstate of a Hamiltonian (so that the phase only depends on the number of periods the beam is left before interference).

Assuming that there is a prescription by which this phase change can be performed, the above prescription can be used to measure the off-diagonal elements of the decoherence functional (see Fig. 3).

Let us assume we have a source S preparing particles in a state $|\psi\rangle$. After exiting S, the beam enters a beam splitter B.S. One of its components then enters a sequence of filters $\alpha_{t_1} \ldots \alpha_{t_n}$ and the other a sequence of filters $\beta_{t'_1} \ldots \beta_{t'_m}$, before they are recombined at C. The beam then propagates to a screen, where its intensity is measured. Now we repeat this experiment many times, but at each time the second component of the split beam has to pass through P.O. which performs the operation of phase change $|\psi\rangle \to e^{i\chi}|\psi\rangle$. Repeating the experiment for different values of χ, we get a function $I(\chi)$, whose maximum determines a phase that is the argument of the value of the decoherence functional between the histories $(\alpha_{t_1} \ldots \alpha_{t_n})$ and $(\beta_{t'_1}, \ldots, \beta_{t'_m})$. The modulus of the phase of the decoherence functional can easily be determined by the maximum value of I (see, Anastopoulos, 2002, for details).

In fact, it can be shown that the decoherence functional contains all information that can be obtained from experiments measuring either probabilities or relative phases. The name decoherence functional is then, rather misleading. Gell-Mann and Hartle introduced it as the object that provides information about decoherence of histories. However, its most important function is that it contains the information of the relative phases, hence of coherence of histories. For this reason a more adequate name for it would be the coherence functional or the phase functional.

It can also be shown that the complex-valued temporal correlation functions can in principle be determined by the measurement of a sufficiently large number of interference phases as described above. In effect, if one can determine the off-diagonal elements of a decoherence functional between a history $\alpha_{ij} = (P_i, t_1; P_j, t_2)$ and the trivial history $\beta = 1$, then for the observable $A = \sum_i a_i P_i$ the time-ordered two-point function will read

$$\langle A_{t_1} A_{t_2} \rangle_Q = \sum_{ij} a_i a_j d(\alpha_{ij}, 1), \tag{3.2}$$

This gives an operational scheme for the determination of the quantum correlation functions through measurements analogous to the one performed for the Pancharatnam phase. The reader is referred to Anastopoulos (2001a) for details.

4. A FRAMEWORK FOR QUANTUM PROCESSES

4.1. Phases as Primitive Ingredients

Usually any discussion of the principles of quantum theory consider probabilities as the basic objects that are predicted by the formalism. But, as we showed there is good reason to consider (also) the phase as primitive ingredients of the formalism. We can then attempt to write an axiomatic scheme that achieves this.

In that case, we must accept that the probabilities of the corresponding quantum theory are nonadditive, hence do not satisfy the Kolmogorov axioms. Accepting nonadditive probabilities has important consequences for the structure of the resulting quantum theory. The theorems of Bell (1964) and Kochen–Specker (1967) that forbid hidden variable theories of reproducing the predictions of quantum theory assume that the corresponding hidden variable theories are either deterministic or stochastic. They do not forbid hidden variable theories that are modeled by a statistical theory that is not described by Kolmogorov probability.

Hence, by accepting phases as primitive ingredients of our formalism, we might be able to write a theory that reproduces the predictions of quantum mechanics, while having observables like any classical theory, purely commutative objects. This is indeed possible and we showed that in reference (Anastopoulos, 2001a) by simply employing the Wigner transform on the standard quantum

mechanical decoherence functional. In other words, the Hilbert space structure and the corresponding non-Boolean (and nondistributive) nature of the events (usually referred to as quantum logic) *is not necessary in a theory that takes phases as primitive ingredients*.

Such a theory needs a mathematical structure analogous to that of a decoherence functional over a classical sample space. We shall explain in detail how such theories are formalized and constructed: we shall call them theories of *quantum processes*, because their mathematical structure is in many respects analogous to the theory of stochastic processes.

We will not attempt to write any interpretation of quantum theory different from Copenhagen; we shall take a strictly operational stance and simply consider that the theories of quantum processes refer only to measurement situations and the probabilities and relative phases always make reference to ensembles of individual quantum systems. We should, however, point out that our longer term perspective is realist: we want to find a way to talk about the physics of individual quantum system. The present is not, however, a good moment for this purpose.

Choosing the sample space: If we want to write a theory that reproduces the results of quantum mechanics, while having a classical sample space, we need to specify what this sample space would be. One could take the stance that the proper quantum mechanical sample space consists of variables very different from the ones that are naturally apparent to us: the true degrees of freedom refer to a subquantum level of reality, quite removed from standard physics (as for instance in 't Hooft, 1999).

We shall take a more conservative approach here. He shall consider that the sample space is essentially the phase space of the corresponding classical system. There are three reasons for our choice:

1. We can show that a theory of quantum processes on the classical phase space allows us to fully reproduce the predictions of standard quantum theory (this is an a posteriori argument).
2. Symplectic manifolds have a very rich geometric structure, that allows us to reproduce many classic quantum mechanical results that seem to need a priori the notion of the Hilbert space. Such is the case for Wigner's classification of particles (from the geometric quantization of Konstant–Souriau (Souriau, 1997)) and many aspects of the spin-statistics relation (Anastopoulos, 2001b).
3. One of our motivation for undertaking this line of research is the attempt to write a quantum theory that has a pronounced spacetime character. In the closely related scheme of histories quantization, Savvidou showed that in the space of *phase space histories* one can always implement a symplectic action of the group of spacetime diffeomorphisms (Savvidou, 2001) (in the case of general relativity it coexists with the algebra of constraints obtained by the $3 + 1$ decomposition). This is a very important part for any quantum

theory that wants to manifest the principle of general covariance and holds only for phase space histories (not configuration space ones).

These are the reasons that makes us consider the phase space histories as defining the basic sample space of quantum theory. Our arguments are sufficient to establish the naturality of our choice; of course, they are not necessary as they are directed by choices related to our long-term aims. In any case, the formalism we shall present makes sense for any possible sample space.

4.2. The Basic Axioms

4.2.1. Sample Space, Events, Observables

At the level of observables, the structure of our theory is identical with that of classical probability theory. That is, we assume the existence of a space Ω of elementary alternatives. A point of Ω corresponds to the most precise information one can extract from a measurement of the quantum system. Note, that at this level we do not distinguish, whether Ω refers to properties of a systems at one moment of time or to histories. Our definitions are general and only later shall we specify the history content.

This space Ω has to be equipped with some additional structure. In general, a measurement will yield some information stating that the system was found in a given subset of Ω. But not all subsets of Ω are suitable to incorporate measurement outcomes. For instance, when we consider position it is physically meaningless to consider the subset of rational values of position (with respect to some unit). One, therefore needs to choose a family of subsets $\mathcal{C}$ of Ω, that correspond to the coarse-grained information we can obtain about the physical systems. These sets are often called *events*. The family $\mathcal{C}$ containing the events has to satisfy some natural mathematical conditions

i. $\Omega \in \mathcal{C}$: if an experiment is performed one of the outcomes will occur.
ii. $\emptyset \in \mathcal{C}$: it is impossible that no outcome results if an experiment is performed.
iii. If $A \in \mathcal{C}$, then $\Omega - A \in \mathcal{C}$: if A is a possible measurement outcome then so can be its complement.
iv. If A, $B \in \mathcal{C}$, then $A \cup B \in \mathcal{C}$ and $A \cap B \in \mathcal{C}$: unions and intersections of experimental outcomes are also possible experimental outcomes.
v. For countably many $A_n \in \mathcal{C}$, $n = 1, 2, \ldots$, $\cup_{n=1}^{\infty} \in \mathcal{C}$. This is a technical condition particularly relevant when dealing with the case where Ω is a manifold.

Equipping Ω with a choice of events turns it into a *measurable space*.

Since in experiments we eventually come to measure real numbers (or occasionally integers, which can be embedded into the real numbers) the mathematical

object that would represent the notion of observable is a map from Ω to $\mathbf{R}$. However, not all possible maps will do: the structures corresponding to measurable sets have to be preserved. Such functions are called *measurable* and in the language of probability theory are known as *random variables*. We shall denote the space they belong to in as $F(\Omega)$.

Among all functions, important are characteristic functions of the various subsets of Ω. These are defined as

$$\chi_A(x) = 1,\ x \in \Omega \tag{4.1}$$

$$= 0,\ x \notin \Omega \tag{4.2}$$

An important property of the characteristic functions is the following. If λ is a possible value of a random variable f and $A_\lambda = f^{-1}(\lambda)$, then it is evident

$$f = \int d\lambda \lambda \chi_{A_\lambda}. \tag{4.3}$$

4.2.2. The Decoherence Functional

A *decoherence functional* Φ[6] is a map from $\mathcal{C} \times \mathcal{C} \to \mathrm{C}$, such that the following conditions are satisfied

B1. *Null triviality*: For any $A \in \mathcal{C}$, $\Phi(\emptyset, A) = 0$.
In terms of our interpretation of the off-diagonal elements of the decoherence functional as corresponding to Pancharatnam phases, there can be no phase measurement if one of the two beams that have to be interfered is absent.

B2. *Hermiticity*: For $A, B \in \mathcal{C}$, $\Phi(B, A) = \Phi^*(A, B)$.
Clearly the phase difference between two histories becomes opposite if we exchange the sequence, by which these histories are considered.

B3. *Positivity:* For any $A \in \mathcal{C}$, $\Phi(A, A) \geq 0$.
This amounts to the fact that the diagonal elements of the decoherence functional are interpreted as probabilities (albeit nonadditive). Operationally probabilities are defined by the number of times a particular event occurred in the ensemble and by definition they can only be positive.

B4. *Normalization:* $\Phi(\Omega, \Omega) = 1$.
Clearly, if no measurement takes place the intensity of the beam would never change.

B5. *Additivity:* If $A, B, C, \in \mathcal{C}$, and $A \cap B = \emptyset$, then $\Phi(A \cup B, C) = \Phi(A, C) + \Phi(B, C)$.

[6] Note that we changed notation for the decoherence functional from d, which is the standard in the bibliography to Φ and was employed in the previous sections. This was done for reasons of notational convenience (d tended to be confused with differentials). The letter Φ stands for phase.

There is no intuitive operational reason, why this should be the case. This property is equivalent to the superposition principle of quantum theory and we can consider that it is forced upon us by experimental results. Of course, this is the property that makes the decoherence functional the natural object to use.

B6. *Boundedness:* For all $A, B \in \mathcal{C}$, $|\Phi(A, B)| \leq 1$.

This arises from the operational procedure for the determination of $|\Phi(A, B)|$ (Anastopoulos, 2002).

These axioms are an adaptation of the axioms written by Isham and Linden (1994) for the case of consistent histories.

There are two points one needs to make regarding these axioms. First, the properties of Φ are identical to the matrix elements of a density matrix over some continuous basis (if this is identified with Ω). So *the difference of quantum from stochastic processes is that they have a density matrix rather than a probability measure over the sample space of histories*.

The second point is that the properties of the decoherence functional are these of a complex probability measure on $\Omega \times \Omega$. So we need not construct any different mathematics for the development of the theory: standard measure theory will suffice. In particular, we can use the Radon–Nikodym and Kolmogorov theorems that are very important in classical probability theory. Since Φ acts on events its action can be extended to all measurable functions on Ω, i.e. we can write it as a map $\Phi : F(\Omega) \otimes F(\Omega) \to \mathcal{C}$.

Now, if on Ω there is a Lebesque measure dx, we can write Φ in terms of s density v, i.e.

$$\Phi(dx, dx') = v(x, x')\, dx\, dx'. \tag{4.4}$$

In the trivial case that Ω refers only to a moment of time, it is easy to see that v is written in terms of the density matrix as

$$v(x, x') = \rho(x, x')\delta(x, x') \tag{4.5}$$

in a continuous basis (like position) or

$$v(x, x') = \langle z|\rho|z'\rangle\langle z'|z\rangle, \tag{4.6}$$

in an overcomplete basis like the coherent states.

4.3. Quantum Processes

We are interested in the nontrivial case, whenever Ω is a space of histories, i.e. it is a suitable chosen subset of $\times_t \Gamma_t$. We shall assume t to take values in some interval of the real line, or the real line itself. An element of Ω will then be a path on Γ and will be written written as $z(\cdot)$.

Given a function f on Γ, we can define a family of functions F_t on Ω indexed by t as

$$F_t[z(\cdot)] = f(z(t)). \tag{4.7}$$

In analogy with the definition of a stochastic process, a *quantum process* is defined as a triple (Ω, Φ, F_t^a), where

- Ω is a path space
- Φ is a decoherence functional satisfying the relevant axioms
- F_t^a is a selected family of observables indexed by t (not necessarily of the type (4.7)).

From the specification of the family of observables F_t^a we can define the mixed correlation functions $G^{n,m}$ as

$$\begin{aligned} &G^{n,m}(a_1, t_1; a_2, t_2; \ldots; a_n, t_n | b_1, t_1'; b_2, t_2'; \ldots; b_m, t_m') \\ &\quad = \Phi\left(F_{t_1}^{a_1} F_{t_2}^{a_2} \ldots F_{t_n}^{a_n}, F_{t_1'}^{b_1} F_{t_2'}^{b_2} \ldots F_{t_m'}^{b_m}\right). \end{aligned} \tag{4.8}$$

Here $G^{n,0}$ are the time-ordered correlation functions, $G^{0,m}$ are the anti-time-ordered and $G^{n,m}$ ones containing mixed entries.

From the hierarchy $G^{n,m}$ of correlation functions associated to F_t^a we can define the corresponding generating functional $Z_F[J_+, J_-]$, which is written in terms of the sources $J_+^a(t)$, $J_-^a(t)$ as

$$\begin{aligned} Z_F[J_+, J_-] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{i^n (-i)^m}{n!m!} \times \sum_{a_1,\ldots,a_n} \sum_{b_1,\ldots,b_m} \int dt_1 \ldots dt_n \, dt_1' \ldots dt_m' \\ \times G^{n,m}(a_1, t_1; \ldots; a_n, t_n | b_1, t_1'; \ldots; b_m, t_m') \\ \times J_+^{a_1}(t_1) \ldots J_+^{a_n}(t_n) J_-^{b_1}(t_1') \ldots J_-^{b_m}(t_m') \end{aligned} \tag{4.9}$$

This is known as the closed-time-path (CTP) generating functional. It was first introduced by Schwinger (1961) and Keldysh (1964). It is particularly relevant to the discussion of systems, in which there does not exist the symmetry of time translation (open quantum systems, field theory in curved spacetime etc). Clearly the CTP generating functional can be written as

$$Z_F[J_+, J_-] = \Phi(e^{iF \cdot J_+}, e^{-iF \cdot J_-}), \tag{4.10}$$

where $F \cdot J_\pm := \int dt \sum_i F_t^i J_\pm(t)$.

Defining Φ, Ω is in general a path space. In order to define Φ on it, we treat it as a complex measure and employ an analogue of *Kolmogorov's theorem* for probability measures on path spaces. Namely, that Φ is uniquely determined by

the discrete-time versions of the decoherence functional: i.e. by a hierarchy of distribution functions

$$\nu^{n,m}(z_1, t_1; z_2, t_2; \ldots; z_n, t_n | z'_1, t'_1; z'_m, t'_m) \tag{4.11}$$

The functions of this hierarchy need to satisfy the *(Kolmogorov) additivity condition*:

$$\int dz_{n+1} \nu^{n+1,m}((z_1, t_1; z_2, t_2; \ldots; z_n, t_n; z_{n+1}, t_{n+1} | z'_1, t'_1; z'_2, t'_2; \ldots; z'_m, t'_m)$$
$$= \nu^{n,m}(z_1, t_1; z_2, t_2; \ldots; z_n, t_n | z'_1, t'_1; z'_2, t'_2; \ldots; z'_m, t_m). \tag{4.12}$$

The point is that a measure (hence a decoherence functional) for the continuous-time case can be determined by the specification of only discrete-time expressions.

4.4. The Kinematical Process

One can construct quantum processes starting from standard quantum theory, by employing the coherent states. In general, the coherent states provide a from a symplectic manifold Γ to the projective Hilbert space: $z \in \Gamma \to |z\rangle\langle z|$. If f is a function on Γ then one can define a corresponding operator on the Hilbert space as

$$A = \int dz f(z) |z\rangle\langle z|. \tag{4.13}$$

This is not the only choice, but as we shall see it is the most natural.

We want first to define the *kinematical process*, i.e. the quantum process that corresponds to a system with vanishing Hamiltonian.

Having Γ one can define the space Ω of continuous paths on Γ and the family of functions F^a_t associated to the f^a of Eq. (4.13). All that is missing from the definition of a quantum processes is the specification of a decoherence functional. This is achieved by specifying the hierarchy of ordered distribution functions $\nu^{n,m}$. To do so, we write the time instants in terms of their ordering $t_1 \leq t_2 \leq \ldots t_n$, and $t'_1 \leq t'_2 \leq \ldots \leq t'_m$. If we write $\hat{\alpha}_z = |z\rangle\langle z|$ we will have

$$\begin{aligned} &\nu^{n,m}_{z_0}(z_1, t_1; z_2, t_2; \ldots; z_n, t_n | z'_1, t'_1; z'_2, t'_2; \ldots; z'_m, t'_m) \\ &\quad = Tr(\hat{\alpha}_{z_n} \hat{\alpha}_{z_{n-1}} \ldots \hat{\alpha}_{z_2} \hat{\alpha}_{z_1} \hat{\alpha}_{z_0} \hat{\alpha}_{z'_1} \ldots \hat{\alpha}_{z'_{m-1}} \hat{\alpha}_{z'_m}) \\ &\quad = \langle z'_m | z_n \rangle \langle z_n | z_{n-1} \rangle \ldots \langle z_2 | z_1 \rangle \langle z_1 | z_0 \rangle \langle z_0 | z'_1 \rangle \langle z'_1 | z'_2 \rangle \ldots \langle z'_{m-1} | z'_m \rangle. \end{aligned} \tag{4.14}$$

Let us now note the following concerning the kinematical process.

1. The expression for the distribution function factorizes in products of the form $\langle z|z'\rangle$. The knowledge of this inner product, suffices to fully determine the kinematical process. In fact, the distribution function $v^{n,m}$ is known as the $n+m+1$ Bargmann invariant (Mukunda and Simon, 1993).
2. The distributions $v^{n,m}$ do not depend on the values of time t, only on their ordering. The same is true for t'. More than that, if we consider the following cyclic ordering for the time instants $t_0 \to t_1 \to t_2 \to \ldots \to t_n \to t'_m \to \ldots \to t'_2 \to t'_1 \to t_0$, the distributions are invariant if we consider any time as origin and then proceed cyclically along the arrows. In other words, the kinematic process manifests the symmetry of a *closed time path*.
3. Unlike stochastic processes, in which the kinematical process is trivial (the hierarchy of distribution functions consists only of products of delta functions), quantum processes manifest all quantum mechanical behaviour (interferences etc) already at the kinematical level. The introduction of dynamics requires only the minor modification of substituting $\langle z|e^{-iH(t-t')}|z\rangle$ $\langle z|z'\rangle$ in Eq. (4.14).
4. Let us consider that the process being defined in the time interval $[0, \tau]$ and consider the distribution function $v^{n,m}$ for large values of n and m. Take for simplicity $n = m = N$. Choose also the time instants such that $|t_i - t_{i-1}| \leq \delta t = \tau/N$ for all i and similarly for t'. Also, let $t_n = t'_m = \tau$. Then we have a discretised approximation to a decoherence functional for continuous paths $z(\cdot)$, $z'(\cdot)$ which for $N \to \infty$ would converge to

$$\Phi(z(\cdot), z'(\cdot)) = e^{-i\int_C \langle z|d|z\rangle} + O(\delta t^2) = e^{i\int_C A} + O(\delta t^2), \qquad (4.15)$$

where C is the closed path obtained by appending the path $z'(\cdot)$ with reverse orientation at the end of $z(\cdot)$. The distribution function for the decoherence functional then converges at the large N limit to the holonomy of a $U(1)$ connection on Γ, the same geometrical object that is introduced in the geometric quantization scheme. Of course, this convergence is to be interpreted with a grain of salt as the support of the decoherence functional is primarily not on differentiable paths, for which the holonomy is rigorously defined.

4.5. Quantum Differential Equations

In any theory that is based on histories, one can sharply distinguish between two different aspects of temporal symmetries. This has been pointed out by Savvidou (1999) and forms one of the basic features of the histories quantization programme. We can define a purely kinematical time translation, by which the

time translation is effected as

$$F_t \rightarrow f_{t+s}, \tag{4.16}$$

and there exists the dynamical time-translation generated by the Hamiltonian. These are completely distinct and correspond to the different functions of the notion of time in the physical theory (causal ordering/kinematics vs. change/dynamics).

As we showed earlier, the characteristic quantum mechanical behavior exists already at the kinematical level and the decoherence functional depend only on the causal ordering of the events constituting the histories. We are, therefore, tempted to take the kinematical process as basic and seek to write a general quantum process in terms of it. In effect, we want to view the kinematical process as the analogue of the Wiener process in classical probability theory: a general object through which any other process can be defined.

In classical stochastic processes the relation between a process x and the Wiener process W[7] is expressed in terms of *stochastic differential equations*, known in physics as Langevin equations. They are, in general, of the form

$$dx^a + f^a(x)\,dt = dW^a(t). \tag{4.17}$$

In effect the Wiener term acts as a random driving force on a deterministic equation. *Can we do the same for the case of the quantum processes?*

It turns out that we can. Using explicitly the distinction between the kinematical and dynamical time translations, we referred to earlier we can write an analogous *quantum differential equation*:

$$dz^a + f^a(z)\,dt = d\xi^a(t), \tag{4.18}$$

where z^a are variables (coordinates on Γ) undergoing a quantum process with a given Hamiltonian H, f^a are some particular functions on phase space depending on the Hamiltonian (Anastopoulos, 2002) and ξ^a are variables on the phase space (the same functions as z^a) but corresponding to a *kinematical process*. As in the classical case, the differentials have t be interpreted with care as in general the sample paths z^a_t are not differentiable. This equation is written at the same level of rigor as the standard stochastic differential equations.

In line with Einstein's remark we gave in section 1, we want to remark on the appealing possibility that Eq. (4.18) can be interpreted as referring to an individual system in analogy to the classical Langevin equations. That is, we can consider that Eq. (4.18) refers to an individual system (a particle), which is found within a "fluctuating environment," that induces the "random forces" $\dot{\xi}^a(t)$. However, these forces are not distributed according to a classical probability distribution, but according to the kinematic processes (and are possibly geometrical in origin).

[7] Denoting the Wiener process as W is a shorthand: the Wiener process refers to the underlying sample space, the particular probability measure and the basic variables W^a.

We are not in a position to argue, whether this interpretation of Eq. (4.18) should be taken seriously or not. The reasons are partly mathematical and partly physical: from the mathematical side we need to verify that such equations are more than empty symbols: is it actually a type of equation that can admit solutions? We hope to justify such equations by adopting the theory of stochastic integrals (of Ito) in the quantum context. From a physical point of view, even though we are committed to finding a description for the individual quantum system, the picture of a particle moving under random forces is not necessarily our first choice: it is too classical and there is no geometric naturality (the functions f in Eq. (4.18) have no apparent geometric interpretation).

Nonetheless, Eq. (4.18) has large theoretical interest: it *demonstrates* that it is possible in principle to unravel the statistical description of quantum theory into a description of individual system. The description in terms of quantum differential equations may not be fundamental, it probably is not the physically correct way to approach individual system, but it *proves a point: a description of individual quantum systems that fully agrees in the ensemble statistics with standard quantum theory is not impossible.*

Moreover, we would like to see, whether it would be possible to simulate its solutions numerically as we can do with stochastic processes. This would provide a way of generating actual trajectories for individual quantum systems.

5. RECONSTRUCTION THEOREM

In the previous section, we showed how to obtain quantum processes starting from quantum theory. Now, we want to invert this procedure and ask how one can obtain standard quantum theory starting from a generic quantum process, that satisfies the axioms stated in section.

Our result is simpler than we expected. We essentially found that we can uniquely determine the quantum mechanical Hilbert space, the observables and the evolution equations from the ingredients of a quantum process, if this process satisfies the (analogue of) the *Markov property*. The Markov property roughly states that if the state of the system (i.e the restricted decoherence functional at a moment of time) is completely specified, then it contains sufficient information to determine the state of the system at any subsequent moment of time.

The Markov property implies (this is often taken as its defining property) that the distribution functions that define decoherence functional can be written as

$$\begin{aligned} &\nu^{N+1,N+1}(z_0, t_0; z_1, t_1; \ldots; z_N, t_{N-1}|z'_0, t_0; z_1, t_1; \ldots z'_{N-1}, t_{N-1}) \\ &\quad = \nu(z_N, z'_N; t_N|z_{N-1}, z'_{N-1}; t_{N-1}) \ldots \nu(z_1, z'_1; t_1|z_0, z'_0; t_0)\rho_0(z_0, z'_0), \quad (5.1) \end{aligned}$$

in terms of a propagator $v(z_1, z_2; t|z_1', z_2'; t')$ and an initial "state" at $t = 0$.[8]

The propagator v needs to satisfy the *quantum Chapman–Kolmogorov equation*:

$$v(z_1, z_1'; t|z_0, z_0'; s) = \int dz\, dz' v(z_1, z_1'; t|z, z'; s')v(z, z'; s'|z_0, z_0'; s). \qquad (5.2)$$

Now, what we have proven is a *reconstruction theorem*, which can loosely be stated as follows.

Reconstruction theorem. *Assume we have a stochastic process* (Ω, Φ, F_t^a)*, that satisfies the Markov property. If in addition*

i. *the propagator is a smooth function of its arguments and the time entries,*
ii. *the process is time-homogeneous,*
iii. *the process is time-reversible,*

then we can reconstruct the quantum mechanical Hilbert space and the Heisenberg evolution equations.

Sketch of the proof: Time homogeneity means that the propagator depends only on the time difference $t - t'$ hence can be written as $v_t(z_1, z_2|z_1', z_2')$. Time-reversibility is defined as $v_t^*(z_1, z_2|z_1', z_2') = v_{-t}(z_1, z_2|z_1', z_2')$. It is easy to show, that this implies that v is factorised as $v_t(z_1, z_2|z_1', z_2') = \psi_t(z_1|z_1')\psi_t^*(z_2|z_2')$, in terms of another kernel that also satisfies a version of the Chapman–Kolmogorov equality

$$\psi_t(z|z') = \int dz''\psi_{t-s}(z|z'')\psi_s(z''|z'). \qquad (5.3)$$

The condition (i) is important. It ensures that when $t \to 0$, ψ remains a nice (i.e. differentiable) function (not a distribution), say $\chi(z|z') := \psi_0(z|z')$) hence *the kinematical process will not be trivial*. In this case the Chapman–Kolmogorov identity states that

$$\chi(z|z') = \int dz''\chi(z|z'')\chi(z''|z'), \qquad (5.4)$$

hence χ defines a projection operator E on $\mathcal{L}^2(\Gamma)$. The range of E is the quantum mechanical Hilbert space H. Moreover, the dynamics encoded in ψ_t correspond to an one-parameter group of unitary transformations that commutes with E and can thus be projected on H giving rise to Hamiltonian evolution.

[8] Note that we have written only the diagonal elements of the hierarchy of functions. But it can be easily shown that they can be used to construct the full hierarchy $v^{n,m}$ by virtue of the Kolmogorov additivity condition.

One can also use a standard *GNS* construction (Klauder and Skagerstam, 1985) to construct a family of coherent states $|z\rangle$ on H such that

$$\langle z|z'\rangle = \chi(z|z'). \tag{5.5}$$

In this case a function f on Γ is mapped into an operator $A = \int dz|z\rangle\langle z|$. In particular, a phase space cell C is mapped to a positive operator $P_C = \int_C dz|z\rangle\langle z|$.□ One remark needs to be made at this point. Our procedure so far is axiomatic and not constructive. If one wants to explicitly construct the quantum process one needs to write the coherent states propagator. This can be rigorously defined by a phase space path integral, in which the Planck's constant enters through a Riemannian metric on the phase space that is employed for regularisation (Klauder, 1988, 1995).

6. INTERPRETATIONAL ISSUES

The difference between quantum processes and standard quantum theory lies only in the determination of which object correspond to sharp events. Quantum mechanics admits projection operators, while the theory of quantum processes admits phase space cells. These are represented by a positive operator-valued-measure $C \to \hat{C} = \int_C dz|z\rangle\langle z|$, for any measurable subset C of Γ.

The question then arises, which of the basic principles of quantum theory is (are) violated by this change and whether this violation has empirical consequences.

It can be shown (Anastopoulos, 2002) that the only difference between the theory of quantum processes and standard quantum mechanics is (what we shall call) the *spectral principle*

The possible values for an observable correspond to the points of the spectrum of the corresponding operator.

It is a corollary of this postulate, that a proposition about possible values of an observable is represented by a projection operator.

Now, in a quantum process the spectrum of an operator is simply not relevant to the values of the corresponding observable, because at the fundamental level observables are functions on the history space Ω. Clearly there is little difference as far as observables with continuous spectrum are concerned (position, momentum etc). The difference lies, of course, in the case of observables with discrete spectrum.

The case of discrete spectrum is, in fact, what has given quantum phenomena their name, as it is this through the discrete spectrum of operators that the paradigmatic quantum behavior is manifested: historically it was the black body radiation, the photoelectric effect and the Bohr's atom transitions that put

discreteness as a basic feature of the new mechanics. For this reason the spectral postulate was highlighted in all early work of quantum theory: it provided a simple solution to the problems that had faced a generation of physicists. Later mathematical development—namely the spectral theorem—offered this postulate the additional justification of mathematical elegance.

It would seem that this is one of the most solid postulates of quantum mechanics, the last one to be taken away from any possible modification of the theory. After all it provides the solution to the physical puzzles that led to quantum mechanics. However, as we are going to argue it is the postulate of quantum theory that is *least* justified empirically, when taken by itself.

To see this we shall consider the case of atom spectroscopy, which has been historically the main arena justifying the spectral postulate. When we study the electromagnetic radiation emitted from atoms, we see that the intensity of the electromagnetic field has peaks in particular discrete values of the frequency. Then assuming energy conservation, the photons are viewed as arising from a transition between two "states" of an atom, each of which is characterized by a sharp value of energy. The fact that we measure a number of sharp peaks rather than a smoother distribution of field intensity plotted versus frequency, leads to the conclusion that the possible values of atom's energy are discrete. If we take that this experiment measures the energy of the atom, then we have discrete values for the energy, something that is naturally explained in terms of the spectral postulate: in any individual measurement only points of the spectrum of the operator are obtained.

We believe that this is a fair summary of the argument that leads to the acceptance of the spectral postulate in this particular context. We shall now see, that the conclusions of the argument is by no means necessary. Let us first make the too obvious remark, that the measurement of the intensity peaks never yields sharp values, rather only peaks with finite width. The width is due not only to experimental errors, but comes fundamentally from the time–energy uncertainty relation. Hence, it is only in an idealization that the atom's energy values are discrete.

However, the most important argument is that the description in terms of atom transitions *is semiclassical rather than quantum*. What we measure in spectroscopy is the energy/frequency of the electromagnetic field. We typically assumed that the emitted photons are incoherent (both in the classical and the quantum sense), so that the emitted electromagnetic field can be considered as an ensemble of photons. Then, we can idealize the experiments as setting filters that allow only very narrow frequency (energy) range to pass and measure the intensities. The whole experiment is then fully described by energy measurements of the photons. One can give an equivalent description in terms of the electromagnetic fields. So the actual observables that correspond to the set-up of the experiment is photon energies or fluxes, *not atomic energies*. And these energies can be described by continuous variables in either quantum theory or in the quantum process description.

The attribution of discrete energy values to the atom comes from a semiclassical *picture* of the atom/field interaction; it involves a mixture of old quantum theory concepts (orbitals, transitions), with the framework of mature quantum theory. This picture is helpful for calculations, it provides an intuitive picture of the interaction, but it is not fundamentally quantum mechanical. A precise treatment ought to consider the combined system fieldatom, interacting perhaps through QED and then consider energy measurements of the electromagnetic field at particular spatial locations. In such a description all information about the process (including the atom's eigenvalues) would be found in the correlation functions of the electromagnetic field: *but these are predicted by quantum processes in full agreement with standard quantum theory*.

What we imply by this argument, is that historically the discrete values of observables actually refer to the spectrum of the Hamiltonian, rather than any arbitrary observable. The information about its eigenvalues is fully contained in the correlation functions: once these are provided, we can read off any discretised behaviour. In other words, *the discrete behavior in quantum theory is not fundamental or ontological, but arises due to particular forms of the dynamics*. This is true even for spin systems: the "discrete" spin values are always measured in conjunction with its coupling to some magnetic field.

7. CONCLUSIONS

Let conclude in the form of a summary:

1. We argued that complex numbers (or a $U(1)$ invariance of probabilities) is an inherent and irreducible component of quantum probability. Their effect is the existence of statistical quantities that can be determined by experiment that have no analogue in quantum probability: these are the geometric phases that can be determined by comparing *two* distinct histories of the system. Compared to classical probability these phases correspond to novel operational concepts.
2. The consistent histories approach provides the best formalism to take the phase information into account: there exists an 1–1 map between observable quantities (including phases) and mathematical objects. This comes from the relation of the values of the decoherence functional to the Pancharatnam phase. We do not need, however, to subscribe to the standard interpretation of consistent histories. Throughout this paper we prefer to keep an operational perspective.
3. Taking phases as primitive objects of the formalism necessitates the use of nonadditive probabilities. Theories with nonadditive probabilities can be described by commutative observables (or classical logic or hidden variables) without violating Bell's theorem or Kochen–Specker's theorem.

4. We can write a theory of quantum processes in analogy with the theory of stochastic processes. This theory has a well-defined sample space[9] (we choose the classical phase space). The only difference is that in the quantum case the relevant object is a "density matrix" in the space of paths. In this picture, the Schrödinger equation is an exact analogue of the Fokker–Planck equation. Moreover, we can unravel the statistical description to write (at least formally) quantum equations. They can be thought of, as corresponding to individual systems; whether they correspond to real physics is doubtful, but they prove that Einstein's suggestion that *quantum theory is a statistical theory arising out of yet unknown physics for the individual system* is possible and not in conflict with any predictions of quantum theory.
5. Its physical implications aside, it would be very interesting to see if the solutions to quantum differential equations can be found numerically. We would be, then, able to simulate the evolution of individual quantum systems.
6. Starting from quantum processes we get standard quantum theory, by assuming the Markov property, time homogeneity, time-reversibility, and the nontriviality of the kinematical process. Hence, the structure of the Hilbert space necessitates the Markov condition, which presupposes a background causal structure. In absence of this *the Hilbert space is not necessary or even natural*. This could be the case in *quantum gravity*.
7. Finally, we want to identify, how the spacetime symmetries are implemented in the theory of stochastic processes: the relation with the histories quantization programme guided our choice of sample space. After all, our motivation is to find a covariant description of quantum systems, that would allow us to tackle the quantization of gravity. What we would like to see is that the complex phases of quantum theory, would be deeply intertwined with the spacetime structure (perhaps in a fashion analogous to an old conjecture by Penrose?).

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[9] In a recent paper, Kent makes the very accurate remark that the quantum measurement problem should be more accurately phrased as what is exactly the sample space of the quantum system (Kent, 2002). Our answer could have come out of Bohr: the sample space is identical to the one of the corresponding classical system.

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